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ISOMETRIC IMMERSIONS OF RIEMANNIAN PRODUCTS IN EUCLIDEAN SPACE

S. ALEXANDER & R. MALTZ

1. Introduction

Consider a riemannian product $M = M_1 \times \cdots \times M_k$ of k connected complete riemannian manifolds, each of which is nonflat, that is, has some nonvanishing sectional curvature. Let $n_i \geq 2$ be the dimension of M_i. J. D. Moore [7] has proved that if the M_i are all compact, then any k-codimensional isometric immersion of M in euclidean space is a product of hypersurface immersions. (The case $k = 2$ was treated in [1].) That is, for any isometric immersion $f: M \to E^N$ if we write $N = (\sum_{i=1}^{k} n_i) + k$, then there exist a decomposition $E^N = E^{n_1+1} \times \cdots \times E^{n_k+1}$ of E^N into the product of k mutually orthogonal subspaces and isometric immersions $f_i: M_i \to E^{n_i+1}$ for which $f(p_1, \cdots, p_k) = (f_1(p_1), \cdots, f_k(p_k))$. The purpose of this paper is to replace compactness with the following condition, which says that no factor M_i contains a "euclidean strip":

(*) No M_i contains an open submanifold which is isometric to the riemannian product $E^{n_i-1}x(-\varepsilon, \varepsilon)$.

Thus the main theorem may be stated as follows. Throughout the paper we assume all structures are C^∞, and use "manifold" to mean connected manifold.

Theorem. *Let $M_1, \cdots, M_k$ be complete nonflat riemannian manifolds satisfying condition (*). Then any k-codimensional isometric immersion of the riemannian product $M = M_1 \times \cdots \times M_k$ in euclidean space is a product of hypersurface immersions.*

An example will be given in § 4 showing that condition (*) cannot be omitted.

It is known that if $M = M_1 \times \cdots \times M_k$ is a riemannian product of complete nonflat riemannian manifolds, and $f: M \times E^{n_0} \to E^{N+n_0}$ is an isometric immersion of codimension k, then f must be trivial on the euclidean factor [6]. That is, there exist an orthogonal decomposition $E^{N+n_0} = E^N \times E^{n_0}$ and an immersion $\bar{f}: M \to E^N$ for which $f(p, p_0) = (\bar{f}(p), p_0)$; such a map is described as "n_0-cylindrical". The following corollary of the main theorem is immediate.

Corollary 1. *Let $M_1, \cdots, M_k$ be complete nonflat riemannian manifolds*

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satisfying condition (*). *Then any k-codimensional isometric immersion of* $M_1 \times \cdots \times M_k \times E^{n_0}$ *in euclidean space may be expressed as a product of hypersurface immersions of the* M_i *and the identity map on* E^{n_0}.

The proof of the main theorem takes as its starting point Moore's elegant solution of the compact problem. Moore's theorem actually states that, given $M = M_1 \times \cdots \times M_k$ where the M_i are complete and of dimension $n_i \geq 2$, and given a k-codimensional isometric immersion $f: M \to E^N$, then f is a product of hypersurface immersions unless M contains a complete geodesic which is mapped by f onto a straight line in E^N. Clearly no such geodesic exists if M is compact. No curvature requirement is stated because it turns out that if f maps no geodesic onto a line, then no M_i is flat.

For any p in M, let $M_i(p)$ denote the copy of M_i in M passing through p. Our proof will show, assuming all M_i nonflat, that f is a product of hypersurface immersions unless $M_i(p)$, for some i and p, contains an open subset U isometric to $(-\varepsilon, \varepsilon) \times E^{n_i - 1}$ on which f acts $(n_i - 1)$-cylindrically. That is, U is foliated by complete totally geodesic hypersurfaces which are carried onto parallel $(n_i - 1)$-planes in E^N. Note that by the Toponogov and Cheeger-Gromoll splitting theorems [10], [2], no such hypersurfaces can exist if M_i has nonnegative sectional curvature or, more generally, nonnegative Ricci curvature. Thus our theorem in combination with the Sacksteder immersion and rigidity theorems for convex hypersurfaces [8], [9] gives

Corollary 2. *For* $1 \leq i \leq k$, *let* M_i *be a complete nonflat riemannian manifold of nonnegative sectional curvature.* (a) *Then any k-codimensional isometric immersion of* $M = M_1 \times \cdots \times M_k$ *in euclidean space* E^N *is a product of k imbeddings of convex hypersurfaces.* (b) *If, further, each* M_i *has a point at which the conullity index of curvature of* M_i *is at least* 3, *then any two isometric immersions of M in* E^N *differ by an isometry of* E^N.

2. Nullity and relative nullity

Suppose $f: M \to E^N$ is any isometric immersion of a riemannian manifold. We view the second fundamental form of f at $p \in M$ as a symmetric vector-valued bilinear form $T: M_p \times M_p \to M_p^{\perp}$, where M_p denotes the tangent space of M at p. The notations $T(x, y)$ and $T_x y$ will be used interchangeably, according to convenience.

Curvature of M is determined by the second fundamental form of f according to the Gauss equation

$$\langle R_{xy}u, v\rangle = \langle T_x u, T_y v\rangle - \langle T_x v, T_y u\rangle \,, \qquad x, y, u, v, \in M_p \,.$$

The *relative nullity space of* f at p is defined by $R_p = \{x \in M_p : T_x y = 0$ for all $y \in M_p\}$. The Gauss equation implies that R_p is contained in the *nullity space of* M at p, defined by $N_p = \{x \in M_p : R_{xy} = 0$ for all $y \in M_p\}$. The dimensions of R_p and N_p are denoted by $\nu(p)$ and $\mu(p)$ respectively.

The nullity and relative nullity spaces were first defined by Chern and Kuiper [3], who showed that the Gauss equation implies

$$0 \leq \mu(p) - \nu(p) \leq k ,$$

where k is the codimension of the immersion. In this paper, we will use a sharpened inequality, namely,

$$(1) \qquad 0 \leq \mu(p) - \nu(p) \leq k - i(p) ,$$

where $i(p)$ denotes the maximum number of mutually orthogonal subspaces of the orthogonal complement $N_p^\perp$ in M_p which are invariant under the action of the curvature transformations R_{xy} for all x and y in M_p. (Our applications will be to the case where $M = M_1 \times \cdots \times M_k$ and $p = (p_1, \cdots, p_k)$, with $i(p)$ being replaced by the number of factors M_i such that M_i has some non-vanishing sectional curvature at p_i.)

The inequality (1) is a consequence of the following lemma.

Lemma 1. *Suppose a riemannian manifold M is isometrically immersed in E^N. If for some p in M, S is a subspace of M_p satisfying the conditions $R_{xy} = 0$ for all x, y in S and $S \cap R_p = 0$, then the dimension of S does not exceed the codimension of the immersion.*

Proof. It will suffice to show the existence of a vector $u \in M_p$ such that the restriction of T_u to S is an injection of S into $M_p^\perp$.

Suppose, to the contrary, that for a given $u \in M_p$ such that the restriction of T_u to S has maximal rank, there is some nonzero $x \in S$ satisfying $T_u x = 0$. Since $x \notin R(p)$, there exists $v \in M_p$ satisfying $T_v x \neq 0$. Furthermore, for any $y \in S$ the Gauss equation gives $0 = \langle T_x u, T_y v \rangle = \langle T_x v, T_y u \rangle$, since $R_{xy} = 0$. This means that for any $t \neq 0$, the nonzero vector $T_{u+tv} x = t T_v x = t T_x v$ lies in $T_{u+tv}(S)$ and is perpendicular to $T_u(S)$. For t sufficiently small, it follows that the dimension of $T_{u+tv}(S)$ exceeds the dimension of $T_u(S)$, in contradiction to the choice of u. q.e.d.

Now to prove the inequality (1), take S to be the subspace of M_p spanned by the $[\mu(p) - \nu(p)]$-dimensional subspace $N_p \cap R_p^\perp$ and nonzero vectors x_i, $1 \leq i \leq i(p)$, one from each invariant subspace of $N_p^\perp$. $R_{x_i x_j} = 0$ follows from $\langle R_{x_i x_j} u, v \rangle = \langle R_{uv} x_i, x_j \rangle = 0$ for all u, v in M_p.

In the two lemmas which follow, we summarize some important facts about nullity and relative nullity which will be needed later. Lemma 2 may be found in [5]. Lemma 3 was proved by P. Hartman in [4].

Lemma 2. *Suppose a riemannian manifold M contains an open subset W on which the nullity spaces N_p have constant dimension $\mu(p) = c$. Then the distribution N on W is completely integrable and the integral submanifolds are totally geodesic in W. Suppose $\gamma\colon [a, b] \to M$ is a geodesic satisfying $\gamma(s) \in W$ and $\gamma'(s) \in N_{\gamma(s)}$ for all $s \in (a, b)$. Then $\mu(\gamma(a)) = \mu(\gamma(b)) = c$, and the nullity spaces are parallel along $\gamma | [a, b]$.*

Lemma 3. *Suppose that an isometric immersion $f: M \to E^N$ of a riemannian manifold M is such that M contains an open subset W on which the relative nullity spaces R_p have constant dimension $\nu(p) = c$. Then the distribution R on W is completely integrable, and the integral submanifolds are totally geodesic in W. Suppose $\gamma: [a, b] \to M$ is a geodesic satisfying $\gamma(s) \in W$ and $\gamma'(s) \in R_{\gamma(s)}$ for all $s \in (a, b)$. Then $\nu(\gamma(a)) = \nu(\gamma(b)) = c$, and the relative nullity spaces are parallel along $\gamma | [a, b]$.*

We turn now to the case of an isometric immersion f of a riemannian product $M = M_1 \times \cdots \times M_k$ in some euclidean space. For fixed $p = (p_1^0, \cdots, p_i^0, \cdots, p_k^0) \in M$, let $M_i(p)$ be the copy $\{(p_1^0, \cdots, p_i, \cdots, p_k^0): p_i \in M_i\}$ of M_i through p. π_i will denote orthogonal projection of M_p onto its subspace tangent to $M_i(p)$. The subspaces R_{ip} and N_{ip} of $\pi_i M_p$ are respectively defined to be the relative nullity space of $f | M_i(p)$ at p and the nullity space of $M_i(p)$ at p. (Note that the latter is determined by p_i but the former is not.)

Since the curvature transformations R_{xy} of M vanish whenever x and y are tangent to different factors, we easilly obtain

$$N_{ip} = N_p \cap \pi_i M_p , \qquad \oplus_{i=1}^k N_{ip} = \oplus_{i=1}^k \pi_i N_p = N_p .$$

Also, the Gauss equation for $\langle R_{xy}x, y \rangle$ shows that if x and y are tangent to different factors, then whenever $T_x x = 0$ holds, $T_x y = 0$ also holds. From this we may deduce

$$R_{ip} = R_p \cap \pi_i M_p .$$

However, the statement $\oplus_{i=1}^k R_{ip} = \oplus_{i=1}^k \pi_i R_p = R_p$ need not be true. If it is true, we say the relative nullity space R_p *conforms to the product structure of* M. In general, we may only assert

$$(2) \qquad \oplus_{i=1}^k R_{ip} \subseteq R_p \subseteq \oplus_{i=1}^k \pi_i R_p \subseteq N_p$$

with equality holding at the first inclusion if and only if it holds at the second. The third inclusion follows from $R_p \subseteq N_p$ and $\pi_i N_p \subseteq N_p$.

We give a simple example to illustrate these remarks. Let $M_1 = M_2 = E^1$, and isometrically immerse $M = E^1 \times E^1$ in E^3 as a right circular cylinder with the image of the lines $y = x + c$ as generators. Specifically, set $f(x, y) = (\cos \tilde{x}, \sin \tilde{x}, \tilde{y})$ where $\tilde{x} = (x - y)/\sqrt{2}$ and $\tilde{y} = (x + y)/\sqrt{2}$. Then M carries one-dimensional distributions $\pi_1 M_p, \pi_2 M_p$ and R_p tangent to the lines $x = c$, $y = c$ and $y = x + c$ respectively. Thus $R_{ip} = R_p \cap \pi_i M_p = 0$; and the spaces $\oplus R_{ip}, R_p$ and $\oplus \pi_i R_p$ have dimensions zero, one and two respectively.

Finally we state three lemmas due to Moore [7]. The assumption here is that $f: M \to E^N$ is a k-codimensional isometric immersion of some riemannian product manifold $M = M_1 \times \cdots \times M_k$ (not necessarily complete.) For the second fundamental form T of f, we say "$T(x_i, x_j) = 0$ holds at p" if this equation holds for every choice of index pair $i \neq j$ and of vectors $x_i \in \pi_i M_p$, $x_j \in \pi_j M_p$. Similary, "$T(x_1, x_j) = 0$ holds at p" means the equation holds for every choice $j \neq 1$, $x_1 \in \pi_1 M_p$, $x_j \in \pi_j M_p$.

It may be helpful in interpreting the lemmas to represent T by a matrix with entries $T(e_a, e_b) \in M_p^\perp$, $1 \leq a, b \leq N - k$, where $e_1, \cdots, e_{N-k}$ is a basis of M_p which conforms to the product structure of M. The condition $T(x_i, x_j) = 0$ becomes the condition that the only nonzero entries occur in diagonal blocks. Note that a tangent vector $x = \sum_{a=1}^{N-k} x^a e_a$ ($x^a \in \boldsymbol{R}$) is in the relative nullity space R_p if and only if the corresponding linear combination of rows vanishes. The condition $T(x_i, x_j) = 0$ thus clearly implies that the projections $\pi_i x$ are relative nullity vectors whenever x is, that is, that R_p conforms to the product structure of M.

Lemma 4. *If $T(x_i, x_j) = 0$ holds at all $p \in M$, and no M_i is everywhere flat, then f is a product of hypersurface immersions.*

Lemma 5. *For $1 \leq i \leq k$, suppose that M_i is not flat at p_i. Then at $p = (p_1, \cdots, p_k)$ in M, $T(x_i, x_j) = 0$ holds.*

In the following lemma, given an open subset S of $M_1(p)$ we say *q is visible along S* from p if there is a geodesic γ satisfying $\gamma(0) = p$, $\gamma(b) = q$, $\gamma(s) \in S$, and $\gamma'(s) \in R_{1\gamma(s)}$ for $0 \leq s < b$.

Lemma 6. (i) *Let S be an open subset of $M_1(p)$ on which the spaces R_{1p} have constant dimension. If a point at which $T(x_1, x_j) = 0$ holds is visible along S from p, then $T(x_1, x_j) = 0$ holds at p also.*

(ii) *Let S be an open subset of $M_1(p)$ having a neighborhood in M on which $T(x_1, x_j) = 0$ holds. If a point at which $T(x_i, x_j) = 0$ holds is visible along S from p, then $T(x_i, x_j) = 0$ holds at p also.*

3. The main theorem

Suppose $f: M \to E^N$ is a k-codimensional isometric immersion of some riemannian product $M = M_1 \times \cdots \times M_k$. Let X be the open subset of M consisting of points at which $T(x_i, x_j) = 0$ fails. If $p = (p_1, \cdots, p_k)$ is such a point, then Lemma 5 implies that for at least one value of i the factor M_i is flat at p_i. Let $k'(p)$ denote the number of factors M_i flat at p_i. Then the sum of the dimensions of these factors is at least $2k'(p)$, so nullity of M satisfies $\mu(p) \geq 2k'(p)$. On the other hand, relative nullity of f and nullity of M satisfy $0 \leq \mu(p) - \nu(p) \leq k'(p)$, according to (1). Therefore

$$\mu(p) \geq \nu(p) \geq \mu(p) - k'(p) \geq k'(p) > 0 \tag{3}$$

holds at every point of X.

The first step of the proof of the main theorem casts light on the example in § 2.

Proposition. *Suppose $f: M \to E^N$ is a k-codimensional isometric immersion of a complete riemannian product $M = M_1 \times \cdots \times M_k$. Then the relative* *nullity spaces of f conform to the product structure of M unless one of the factors M_i is everywhere flat.*

Proof. Suppose there are points at which the relative nullity spaces R_p do not conform, that is, at which $R_p \neq \oplus_{i=1}^k \pi_i R_p$ holds. Let $X' \subseteq M$ be the open set consisting of all such points.

Since we know $X' \subseteq X$ by the remark proceding Lemma 4, then (3) holds on X'. By letting $V \subseteq X'$ be the minimum set for ν on X', and $W \subseteq V$ be the minimum set for μ on V, we obtain a nonempty open subset W of X' on which the dimensions of the relative nullity spaces and nullity spaces respectively are constant and positive. Let R denote the distribution of relative nullity spaces on W.

Choose any $p \in W$. The leaves of R are totally geodesic in W by Lemma 3, so for a given initial condition $\gamma'(0) \in R_p$ the corresponding M-geodesic γ is tangent to R as long as it remains in W. Suppose $\gamma | [0, b)$ lies in W. Since both R and the distributions tangent to the factors are parallel along $\gamma | [0, b]$, the fact that R_p does not conform to the product structure of M implies that $R_{\gamma(b)}$ does not. That is, $\gamma(b) \in X'$. Since by Lemmas 2 and 3, ν and μ do not change at $\gamma(b)$, we have further $\gamma(b) \in W$. It follows that γ does not leave W, so the leaf through p of R is complete. Next we show that this can only happen if one of the factors is everywhere flat.

Set $k' = k'(p)$, and reorder the factors so that the first k' are flat at p. Now $\oplus_{i=1}^k \pi_i R_p$ lies in the nullity space N_p of M by (2), has dimension larger than the dimension of R_p, and hence has dimension at least $\mu(p) - k' + 1$ by (3). Thus its codimension in N_p is at most $k' - 1$. Since $N_p = \oplus_{i=1}^k \pi_i N_p$, it follows that the codimension of $\oplus_{i=1}^{k'} \pi_i R_p$ in $\oplus_{i=1}^{k'} \pi_i N_p$ is at most $k' - 1$. But we have ordered the factors so that the latter is all of $\oplus_{i=1}^{k'} \pi_i M_p$. It follows that $\pi_i R_p = \pi_i M_p$ for some i, and we may assume $i = 1$.

Thus for any $x_1 \in \pi_1 M_p$, there exists some $x = x_1 + y \in R_p$, where y is orthogonal to $\pi_1 M_p$. Consider the complete geodesic $\gamma = \gamma_1 \times \cdots \times \gamma_k$ in M with initial condition x. γ lies entirely in W because the leaf through p of R is totally geodesic and complete. By Lemma 2, the distribution of nullity spaces $N_{\gamma(t)}$ is parallel along γ, so $\pi_1 M_{\gamma(t)} \subseteq N_{\gamma(t)}$ holds for every value of t because it holds at p. But then M_1 is flat at $\gamma_1(t)$ for every value of t. Since x_1 is arbitrary and γ_1 is a complete geodesic in M_1 with initial condition x_1, it follows that M_1 is everywhere flat.

Proof of main theorem. Let $f: M \to E^N$ be a k-codimensional isometric immersion of $M = M_1 \times \cdots \times M_k$, where the M_i are complete and nonflat, and suppose f is not a product of hypersurface immersions. We wish to show that condition (*) is violated.

By Lemma 4 we know X is not empty, where X still denotes the subset of M on which $T(x_i, x_j) = 0$ fails. We take $W \subseteq X$ to be a connected component of the minimum set for ν on X.

Since the spaces R_p have constant dimension on W and conform to the product structure of M by the preceding proposition, it follows that the spaces R_{ip} have constant dimension on W for each i. This is because each point has a neighborhood on which the dimension of R_{ip} does not increase; and by $R_p = \bigoplus_{i=1}^k R_{ip}$, a decrease in one would force an increase in another.

Repeating an argument from the proof of the proposition, since for any $p_0 \in W$ the codimension of R_{p_0} in N_{p_0} is at most $k'(p_0)$ by (3), then for some i the dimension of $\pi_i R_{p_0}$ is at least $n_i - 1$. Since R_{p_0} conforms to the product structure, $R_{ip_0} = \pi_i R_{p_0}$. Thus, taking $i = 1$, we conclude that W carries a distribution R_1 of dimension either $n_1 - 1$ or n_1, where each R_{1p} is the relative nullity space of $f \mid M_1(p)$. Applying Lemma 3 to the open subset $M_1(p) \cap W$ of any $M_1(p)$ shows that R_1 is integrable and its leaves are totally geodesic in M.

Suppose $T(x_1, x_j) = 0$ holds everywhere on W. For a given p_0 in W, define $S = M_1(p_0) \cap W$. Then for any $p \in S$, Lemma 6 (ii) says that only points of X are visible along S from p. That is, if $\gamma \mid [0, b)$ is any geodesic in S tangent to R_1, where $\gamma(0) = p$, then $\gamma(b)$ lies in X. But since $R_1 \subseteq R$, Lemma 3 says that ν does not change at $\gamma(b)$, so $\gamma(b)$ lies in $S = M_1(p_0) \cap W$ by definition of W. Thus the leaves of the distribution R_1 on S are complete.

The other possibility is that $T(x_1, x_j) = 0$ fails at some $p_0 \in W$. Now define S to consist of all points of $M_1(p_0) \cap W$ at which $T(x_1, x_j) = 0$ fails. Then for any $p \in S$, Lemma 6 (i) says that only points at which $T(x_1, x_j) = 0$ fails are visible along S from p. Then for any geodesic $\gamma \mid [0, b)$ in S tangent to R_1, with $\gamma(0) = p$, we have $\gamma(b) \in X$. $\gamma(b) \in W$ follows as before, hence $\gamma(b) \in S$ by definition of S. Thus again the leaves of the distribution R_1 on S are complete.

In any case, we conclude that some $M_1(p_0)$ contains a nonempty open subset S such that the relative nullity spaces of $f \mid M_1(p_0)$ have dimension $n_1 - 1$ on S, and the leaves of the corresponding distribution R_1 on S are complete. (The possibility that R_1 has dimension n_1 is ruled out because by assumption M_1 is not flat.) It remains to show, assuming S connected, that such leaves must be carried onto parallel $(n_1 - 1)$-planes in E^N. In the remainder of this section we consider only the immersion $f \mid M_1(p_0)$, so we suppress subscripts and from now on write $M = M_1(p_0)$, $f = f \mid M_1(p_0)$, $n = n_1$, and $R = R_1$.

Now let p be a point of S and $L = L(p)$ be the leaf of R through p. Since L is complete, f maps L isometrically onto an $(n-1)$-plane in E^N. Furthermore, for all points $r \in L$ the image n-planes $f_*(M_r)$ are constant. (These assertions are easily verified using the definition of R and the fact that L is totally geodesic in M.) Without loss of generality, assume p is identified with the origin in E^N, L is identified with the $y^1 \cdots y^{n-1}$-plane, and M is tangent to the $y^1 \cdots y^n$-plane at every point of L.

Each $q \in S$ has a neighborhood carrying Frobenius coordinates $\{u^1, \cdots, u^{n-1}, w\}$, that is, coordinates for which the hypersurfaces $w =$ constant are tangent to R. Suppose we know that the translation of $f(L(q))$ to the origin is transverse to the $y^n \cdots y^N$-plane. Then by the inverse function theorem, we may take u^i to be the restriction of $y^i \circ f$, $1 \leq i \leq n-1$. In the following such a coordinate neighborhood will be said to be "adapted", and $y^i \circ f$ will be abbreviated y^i.

Let N_0 be an adapted coordinate neighborhood of p, coordinatized by $\{y^1, \cdots, y^{n-1}, w_0\}$ where $w_0(p) = 0$. Write $L(c)$ for the complete leaf of R passing through the point with coordinates $(0, \cdots, 0, c)$. Then $\{y^1, \cdots, y^{n-1}\}$ is one-one on each $L(c)$, and it follows that the $L(c)$ are all distinct. Thus w_0 extends to a function w on the open set $U = \bigcup_{-\varepsilon<c<\varepsilon} L(c)$. To verify that the one-one map $\{y^1, \cdots, y^{n-1}, w\}$ of U onto $E^{n-1} \times (-\varepsilon, \varepsilon)$ is a diffeomorphism, let $q \in L(c)$ be any point of U. Join $(0, \cdots, 0, c) \in N_0$ to q by a path γ in $L(c)$, and cover γ by finitely many adapted neighborhoods $N_0, \cdots, N_j$, where N_j contains q and carries coordinates $y^1, \cdots, y^{n-1}, w_j$. It suffices to show w varies smoothly with w_j at q and $(\partial w/\partial w_j)(q) \neq 0$. This may be done in j steps; at the first step we have $(\partial w/\partial w_0)(\partial w_0/\partial w_1) = \partial w/\partial w_1$ on $N_0 \cap N_1 \cap \gamma$, and both left hand factors are nonzero.

Now express f on U as a function of the coordinates $\{y^1, \cdots, y^{n-1}, w\}$. Set $A_i(w) = (\partial f/\partial y^i)(0, \cdots, 0, w)$ for $1 \leq i \leq n-1$. Thus $\langle A_i(w), e_j \rangle = \delta_{ij}$, $1 \leq i, j \leq n-1$, where the e_i are part of the standard basis of E^N. It follows that

$$f(y^1, \cdots, y^{n-1}, w) = \textstyle\sum_{i=1}^{n-1} y^i A_i(w) + f(0, \cdots, 0, w) .$$

Furthermore, $\partial f/\partial w$ is always orthogonal to e_i, $1 \leq i \leq n-1$. Since f is an immersion which places M tangent to the $y^1 \cdots y^n$-plane along $L = L(0)$, this means $(\partial f/\partial w)(y^1, \cdots, y^{n-1}, 0)$ is always a nonzero multiple of e_n.

Now to conclude $(dA_i/dw)(0) = 0$, we apply [4, Lemma 4.1] of P. Hartman. The argument is repeated here because in this special case it is very short. We have

$$(\partial f/\partial w)(y^1, \cdots, y^{n-1}, 0) = \textstyle\sum_{i=1}^{n-1} y^i (dA_i/dw)(0) + (\partial f/\partial w)(0, \cdots, 0) .$$

Taking values of 0 and 1 variously for $y^1, \cdots, y^{n-1}$ shows that each $(dA_i/dw)(0)$ is a multiple of e_n. The assumption $(dA_i/dw)(0) \neq 0$ for some i would imply $(dA_i/dw)(0) = c(\partial f/\partial w)(0, \cdots, 0)$ for some $c \neq 0$, and hence $(\partial f/\partial w)(0, \cdots, -c^{-1}, \cdots, 0) = 0$, which is false.

This completes the proof that f carries the leaves of R onto parallel $(n-1)$-planes in E^N. Taking the standard metric on $(-\varepsilon, \varepsilon)$, and taking w to give arc length along the curve $y^1 = \cdots = y^{n-1} = 0$, we find that $\{y^1, \cdots, y^{n-1}, w\}$ is an isometry of U onto the riemannian product $E^{n-1} \times (-\varepsilon, \varepsilon)$. This completes the proof of the theorem.

4. Immersions which are not products

Yeaton Clifton has given, in private communication, a method of constructing a 2-codimensional immersion into euclidean space which induces a parallel line field but does not map the corresponding integral curves into planes. His construction is used below to show that for certain riemannian product manifolds M, an isometric immersion of M which is the product of hypersurface immersions may be continuously deformed through nonproduct isometric immersions of M. In particular, it will follow that condition (*) in the main theorem cannot be omitted.

Let $f_1: M_1 \to E^{n_1+1}$ be any isometric immersion of codimension one, where M_1 is a compact riemannian manifold with metric g_1, and let I denote an interval $(-\varepsilon, \varepsilon)$ with the standard metric. For $N = n_1 + 3$, we have the trivial isometric immersion $M_1 \times I \to E^N$ given by

$$(m, t) \to (\textstyle\sum_{i=1}^{N-2} f_1^i(m)e_i) + te_{N-1}\,, \tag{4}$$

where $e_1, \cdots, e_N$ is the standard basis of E^N.

Now let $\gamma: (-\varepsilon, \varepsilon) \to E^N$ be a regular curve satisfying $\gamma(t) = te_{N-1}$ when $\frac{1}{2}\varepsilon \le |t| < \varepsilon$. We require that γ take its values in the three-dimensional subspace spanned by e_{N-2}, e_{N-1}, e_N, and carry a smooth frame field x_{N-2}, x_{N-1}, x_N satisfying $x_i(t) = e_i$ when $\frac{1}{2}\varepsilon \le |t| < \varepsilon$, and satisfying the Frenet equations:

$$\begin{aligned} \frac{d\gamma}{dt} &= x_{N-1}\,, \qquad & \frac{dx_{N-1}}{dt} &= \kappa x_N\,, \\ \frac{dx_N}{dt} &= \kappa(-x_{N-1} + \alpha x_{N-2})\,, \qquad & \frac{dx_{N-2}}{dt} &= -\kappa\alpha x_N\,. \end{aligned} \tag{5}$$

Here $\alpha(t)$ is a smooth function satisfying $\alpha(t) = 0$ for $\frac{1}{2}\varepsilon \le |t| < \varepsilon$. We also require that for some $t \in (-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon)$ both $\kappa(t)$ and $\alpha(t)$ are nonzero, so that the image of γ does not lie in any two-dimensional subspace.

Define a map $h: M_1 \times I \to E^N$ by

$$h(m, t) = \textstyle\sum_{i=1}^{N-3} f_1^i(m)e_i + f_1^{N-2}(m)[x_{N-2} + \alpha x_{N-1}](t) + \gamma(t)\,.$$

Observe that h agrees with (4) for $\frac{1}{2}\varepsilon \le |t| < \varepsilon$. For local coordinates m^j on M_1 we have

$$\frac{\partial h}{\partial m^j} = \sum_{i=1}^{N-3} \frac{\partial f_1^i}{\partial m^j} e_i + \frac{\partial f_1^{N-2}}{\partial m^j} x_{N-2} + \frac{\partial f_1^{N-2}}{\partial m^j} \alpha x_{N-1}\,, \tag{6}$$

$$\partial h/\partial t = [1 + (d\alpha/dt)f_1^{N-2}]x_{N-1}\,. \tag{7}$$

Since f_1 is an immersion, the matrix $(\partial f_1^i/\partial m^j)$, $i \le i \le N-2$, has maximal rank. Furthermore, because M_1 is compact, we may assume $|d\alpha/dt|$ small enough to ensure that (7) never vanishes, so that h an immersion.

Now define $\tilde{s}: M_1 \times I \to M_1 \times E^1$ by $\tilde{s}(m, t) = (m, s(m, t))$, where $s(m, t) = t + \alpha(t)f_1^{N-2}(m)$. (The effect of this map is to reparametrize each of the curves tangent to $\partial/\partial t$ by h-induced arc length s.) Since $\partial s/\partial t = 1 + (d\alpha/dt)f_1^{N-2} \neq 0$, $\tilde{s}$ is regular. Furthermore, s is strictly monotone in t and satisfies $s(m, t) = t$ for $-\frac{1}{2}\varepsilon \leq |t| < \varepsilon$. It follows that $\tilde{s}$ is a diffeomorphism of $M_1 \times I$ onto $M_1 \times I$.

We claim that $f = h \circ \tilde{s}^{-1}$ is an isometric immersion of the riemannian product $M_1 \times I$ in E^N; that is, that the f-induced metric g is given by $g = g_1 + ds^2$.

First it must be shown that $\partial/\partial s$ is parallel with respect to g. At any point (m, s), we have $\partial f/\partial s = (\partial h/\partial t)(\partial t/\partial s) = x_{N-1}(t)$ for $t = t(m, s)$. Thus $\partial^2 f/\partial s^2 = (dx_{N-1}/dt)(\partial t/\partial s)$ and $\partial^2 f/\partial m^j \partial s = (dx_{N-1}/dt)(\partial t/\partial m^j)$, both of which are parallel to $x_N(t)$ by (5) and therefore orthogonal to the image of f by (6) and (7). Hence the connection induced by f satisfies $\nabla_{\partial/\partial s}\partial/\partial s = 0$ and $\nabla_{\partial/\partial m^j}\partial/\partial s = 0$.

By construction, g and $g_1 + ds^2$ agree for $\frac{1}{2}\varepsilon \leq |s| < \varepsilon$. Furthermore, we have just shown that $\partial/\partial s$ is a parallel unit vector field on $M_1 \times I$ with respect to both metrics. In particular, g is locally a product metric. Cover any s-curve, $-\frac{1}{2}\varepsilon \leq s \leq \frac{1}{2}\varepsilon$, by finitely many g-product neighborhoods. Since s gives arc length in both metrics, it follows that g and $g_1 + ds^2$ agree on a neighborhood of the curve. Thus $g = g_1 + ds^2$ holds everywhere.

Observe, however, that f is not a product of hypersurface immersions. Indeed, for fixed $m \in M_1$, the image of I is tangent to the $x_{N-1}(t)$, which by construction do not lie in any two-dimensional subspace.

Now suppose $f_2: M_2 \to E^{n_2+1}$ is any isometric immersion of codimension one such that M_2 contains an open subset U isometric to $(-\varepsilon, \varepsilon) \times E^{n_2-1}$, and such that f_2 is totally geodesic on U. Let $f_1: M_1 \to E^{n_1+1}$ be as before. The restriction of the product immersion $f_1 \times f_2$ to $M_1 \times U$ is given by

$$(m, (t, r^1, \cdots, r^{n_2-1})) \to \sum_{i=1}^{N-2} f_1^i(m)e_i + te_{N-1} + \sum_{k=1}^{n_2-1} r^k e_{N+k} \, ,$$

where still $N = n_1 + 3$. But we have seen how to construct an isometric immersion f of $M_1 \times U$, leaving the above terms involving r^k unchanged, which is not a product of hypersurface immersions and agrees with $f_1 \times f_2$ whenever $-\frac{1}{2}\varepsilon \leq |t| < \varepsilon$. This means that f and $f_1 \times f_2$ may be pieced together to obtain a 2-codimensional isometric immersion of $M_1 \times M_2$ which is not a product of hypersurface immersions. Finally, we point out that a continuous variation of curves γ about the curve $\gamma_0(t) = te_{N-1}$ gives a continuous variation of such isometric immersions about the product immersion $f_1 \times f_2$.

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University of Illinois, Urbana
Technion Israel Institute of Technology